Available online at www.sciencedirect.com

SCIENCE DIRECT

Journal of Geometry and Physics 51 (2004) 372–386

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Dirac eigenvalue estimates on two-tori☆

Bernd Ammann

Universität Hamburg, Bundesstraße 55, 20 146 Hamburg, Germany

Received 29 March 2002; accepted 26 December 2003

Available online 18 March 2004

Abstract

We prove a lower bound for the eigenvalues of the Dirac operator on two-dimensional tori equipped with a non-trivial spin structure.

MSC: 53C27; 58J50; 53C80

JGP SC: Global differential geometry

Keywords: Dirac operator; Laplace operator; Spectrum; Conformal metrics; Two-dimensional torus; Spin structures

1. Introduction

Friedrich [6] proved, that if the scalar curvature of a compact spin manifold is bounded from below by a positive constant s_0, then any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} s_0.$$

This inequality was improved in case of restricted holonomy, e.g. [11–13]. Another lower estimate for Dirac eigenvalues was proven by Hijazi [9]: the square of any eigenvalue of the Dirac operator is bounded below by the first eigenvalue of the Yamabe operator (conformal Laplacian).

However, for the two-dimensional torus, all these lower bounds are trivial. The two-dimensional torus carries four different spin structures. In general, the spectrum of the Dirac operator will depend on the choice of spin structure. For one of the spin structures,

☆ Partially supported by The European Contract Human Potential Programme, Research Training Networks HPRN-CT-2000-00101 and HPRN-CT-1999-00118.

[1] *E-mail address:* ammann@math.uni-hamburg.de (B. Ammann).
URL: http://www.math.uni-hamburg.de/home/ammann.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2003.12.004

the so-called *trivial* spin structure, zero is in the spectrum, for the other spin structures, it is not.

In the present article we will derive an estimate depending on the spin structure, in order to control the size of the gap in the spectrum around zero.

Let us fix a Riemannian metric and a non-trivial spin structure on T^2.

The *systole* is defined to be the shortest length of a non-contractible loop. Similarly, the *spin-systole* spin-sys_1 is the shortest length of a closed curve along which the spin structure is non-trivial.

We will show (Corollary 2.3) that any eigenvalue λ of the Dirac operator on the torus satisfies

$$\lambda^2 \geq C \frac{\pi^2}{\text{spin-sys}_1^2},$$

where $C > 0$ is an explicitly given expression in the area, the systole and the L^p-norm of the Gaussian curvature, $p \in (1, \infty)$.

The estimate of this paper is an extension of results in [3]. This estimate was the first estimate for Dirac eigenvalues that depends on the spin structure and that holds on manifolds without any symmetry assumptions.

Another estimate for the Dirac eigenvalues on compact oriented surfaces of arbitrary genus has been proven in [1]. This bound depends on different data and uses completely different techniques.

Under suitable curvature conditions the results of the present article yield better estimates for Dirac eigenvalues than [1]. This type of estimate is useful for applications to the Willmore functional [2–4].

2. Main results

Fix a Riemannian metric g and a spin structure χ on the two-dimensional torus T^2. Recall that the L^2-norm of $\alpha \in H^1(T^2, \mathbb{R})$ is $\|\alpha\|_{L_2}^2 := \inf \int |\omega|_g^2 \mathrm{dvol}_g$, where the infimum runs over all smooth one-forms ω representing α. Note that the L^2-norm is invariant under conformal rescaling.

The integer cohomology classes $H^1(T^2, \mathbb{Z})$ are viewed as a lattice in $H^1(T^2, \mathbb{R}) \cong \mathbb{R}^2$. We equip

$$H^1(T^2, \mathbb{Z}_2) \cong \frac{(1/2)H^1(T^2, \mathbb{Z})}{H^1(T^2, \mathbb{Z})}$$

with the quotient norm, i.e. for $\beta \in H^1(T^2, \mathbb{Z}_2)$ we set

$$\|\beta\|_{L_2} := \inf \|\alpha\|_{L_2},$$

where $\alpha \in (1/2)H^1(T^2, \mathbb{Z})$ runs over all representatives of β.

By identifying the trivial spin structure on T^2 with $0 \in H^1(T^2, \mathbb{Z}_2)$ the set of all spin structures is identified with $H^1(T^2, \mathbb{Z}_2)$. Hence $\|\chi\|_{L_2}$ is a well-defined invariant of the spin structure χ and of the conformal type.

Let

$$\sigma_1(T^2, g) := \inf\{\|\alpha\|_{L_2} | \alpha \in H^1(T^2, \mathbb{Z}_2), \alpha \neq 0\},$$

be the *cosystole*.

For the formulation of our statement the following definition is required.

Definition 2.1. For any $p > 1$, let $\mathcal{S}_p$ be the function given by the expression

$$\mathcal{S}_p(\mathcal{K}, \mathcal{K}', \mathcal{V}) := \frac{p}{p-1}\left[\frac{\mathcal{K}'}{4\pi} + \frac{1}{2}\left|\log\left(1 - \frac{\mathcal{K}}{4\pi}\right)\right| + \frac{\mathcal{K}}{8\pi - 2\mathcal{K}}\log\left(\frac{2\mathcal{K}'}{\mathcal{K}}\right)\right] + \frac{\mathcal{K}\mathcal{V}}{8}$$

for $\mathcal{K} \in (0, 4\pi)$, $\mathcal{K}' \in [\mathcal{K}, \infty)$ and $\mathcal{V} \in [0, \infty)$. We extend continuously by setting

$$\mathcal{S}_p(0, \mathcal{K}', \mathcal{V}) := \frac{p}{p-1}\frac{\mathcal{K}'}{4\pi}.$$

In this paper we will prove the following theorem.

Theorem 2.2. *Let (T^2, g) be a Riemannian two-torus with a non-trivial spin structure χ. Assume that $\|\mathcal{K}_g\|_{L^1(T^2,g)} < 4\pi$. Then any eigenvalue λ of D satisfies*

$$\lambda^2 \mathrm{area}(T^2, g) \geq \frac{4\pi^2\|\chi\|^2_{L_2}}{\exp(2\mathcal{S}_p(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}\mathrm{area}(T^2, g)^{1-(1/p)}, \sigma_1(T^2, g)^{-2}))},$$

where $\mathcal{S}_p$ is the function defined in Definition 2.1. Equality is attained for the smallest positive eigenvalue if and only if g is flat.

From this theorem we will derive a corollary estimating λ^2 in terms of the systole sys_1 and the *spin-systole* spin-sys_1

$$\mathrm{sys}_1(T^2, g) := \inf\{\mathrm{length}(\gamma) \mid \gamma \quad \text{is a non-contractible loop.}\},$$
$$\text{spin-sys}_1(T^2, g, \chi) := \inf\{\mathrm{length}(\gamma) \mid \gamma \quad \text{is a loop with} \chi([\gamma]) = -1.\}.$$

Corollary 2.3. *Let (T^2, g) be a Riemannian two-torus with a non-trivial spin structure χ. Assume that $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then any eigenvalue λ of D satisfies*

$$\lambda^2 \text{spin-sys}_1(T^2, g, \chi)^2 \geq \frac{\pi^2}{\exp\left(4\mathcal{S}_p\left(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}\mathrm{area}(T^2, g)^{1-(1/p)}, \frac{\mathrm{area}(T^2,g)}{\mathrm{sys}_1(T^2,g)^2}\right)\right)}.$$

Equality is attained for the smallest positive eigenvalue if and only if:

(a) *g is flat, i.e. (T^2, g) is isometric to $\mathbb{R}^2/\Gamma$ for a suitable lattice Γ*, and
(b) *there are generators γ_1, γ_2 of Γ satisfying $\gamma_1 \perp \gamma_2$, $\chi(\gamma_1) = 1$ and $\chi(\gamma_2) = -1$.*

Remark. Using similar techniques it is possible to obtain similar upper and lower bounds for the first and for all higher eigenvalues, both for the trivial and non-trivial spin structures [2,3].

Proof of the theorem. Because of the uniformization theorem we can write g as $g = \mathrm{e}^{2u} g_0$ with a real-valued function u and a flat metric g_0. This function u solves the Kazdan–Warner equation

$$\Delta_g u = \mathrm{e}^{-2u} \Delta_{g_0} u = K_g.$$

A large part of this paper is devoted to the proof of a Sobolev type inequality, which yields an upper bound for the oscillation $\operatorname{osc} u := \max u - \min u$ (Section 6). We obtain

$$\operatorname{osc} u \leq \mathcal{S}_p(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)} \operatorname{area}(T^2, g)^{1-(1/p)}, \sigma_1(T^2, g)^{-2}). \tag{1}$$

This estimate is optimal if and only if g is flat.

For flat tori the spectrum of the Dirac operator is known: it can be calculated in terms of the dual lattice corresponding to (T^2, g_0) (we recall this in Section 5). As a consequence of this, any eigenvalue λ_0 of the Dirac operator on the flat torus (T^2, g_0, χ) satisfies

$$\lambda_0^2 \operatorname{area}(T^2, g_0) \geq 4\pi^2 \|\chi\|_{L^2}^2. \tag{2}$$

Obviously we have

$$\operatorname{area}(T^2, g) \geq \mathrm{e}^{2\min u} \operatorname{area}(T^2, g_0). \tag{3}$$

Proposition 3.1 now provides the remaining step. There we show that for any Dirac eigenvalue λ on (T^2, g, χ) there is a Dirac eigenvalue λ_0 on (T^2, g_0, χ), such that

$$\lambda^2 \geq \mathrm{e}^{-2\max u} \lambda_0^2. \tag{4}$$

Combining (1)–(4) we obtain the theorem. □

Proof of the corollary. In Section 4 we prove the inequalities

$$\frac{\operatorname{sys}_1(T^2, g)^2}{\operatorname{area}(T^2, g)} \leq \frac{\operatorname{sys}_1(T^2, g_0)^2}{\operatorname{area}(T^2, g_0)} = \sigma_1(T^2, g_0)^2 = \sigma_1(T^2, g)^2,$$

$$\mathrm{e}^{2\operatorname{osc} u} \frac{\operatorname{spin-sys}_1(T^2, g, \chi)^2}{\operatorname{area}(T^2, g)} \geq \frac{\operatorname{spin-sys}_1(T^2, g_0, \chi)^2}{\operatorname{area}(T^2, g_0)} \geq \frac{1}{4\|\chi\|_{L^2(T^2,g_0)}^2} = \frac{1}{4\|\chi\|_{L^2(T^2,g)}^2}.$$

Together with the monotonicity of $\mathcal{S}_p$ in the last argument we obtain the corollary. □

3. Comparing spectra of conformal manifolds

In this section we will compare Dirac eigenvalues on spin-conformal manifolds.

Proposition 3.1. *Let M be a compact manifold with two conformal metrics $\tilde{g}$ and $g = \mathrm{e}^{2u} \tilde{g}$. Let D and $\tilde{D}$ be the corresponding Dirac operators with respect to the same spin structure. We denote the eigenvalues of D^2 by $\lambda_1^2 \leq \lambda_2^2 \leq \cdots$ and the ones of $\tilde{D}^2$ by $\tilde{\lambda}_1^2 \leq \tilde{\lambda}_2^2 \leq \cdots$.*

Then

$$\lambda_i^2 \min_{m\in M} \mathrm{e}^{2u(m)} \le \tilde{\lambda}_i^2 \le \lambda_i^2 \max_{m\in M} \mathrm{e}^{2u(m)}, \qquad \forall i = 1, 2, \ldots$$

Proof. Let $n := \dim M$. We have $\mathrm{dvol}_g = \mathrm{e}^{nu}\mathrm{dvol}_{\tilde{g}}$. There is an isomorphism of vector bundles [5], [10, Satz 3.14] or [9, 4.3.1]

$$\Sigma M \to \tilde{\Sigma} M, \qquad \Psi \mapsto \tilde{\Psi},$$

over the identity id: $M \to M$ satisfying

$$\tilde{D}(\tilde{\Psi}) = \mathrm{e}^u \widetilde{D\Psi} \quad \text{and} \quad |\tilde{\Psi}| = \mathrm{e}^{((n-1)/2)u}|\Psi|. \tag{5}$$

Let $(\Psi_i | i = 1, 2, \ldots)$ be an orthonormal basis of the sections of ΣM with Ψ_i being an eigenspinor of D^2 to the eigenvalue λ_i^2 with respect to the flat metric $\tilde{g}$. The vector space spanned by $\Psi_1, \ldots, \Psi_i$ will be denoted by U_i.

We can bound $\tilde{\lambda}_i^2$ by the Rayleigh quotient

$$\tilde{\lambda}_i^2 \le \max_{\tilde{\Psi}\in U_i - \{0\}} \frac{(\tilde{D}\tilde{\Psi}, \tilde{D}\tilde{\Psi})_{\tilde{g}}}{(\tilde{\Psi}, \tilde{\Psi})_{\tilde{g}}}.$$

Plugging (5) into this expression we conclude $\tilde{\lambda}_i^2 \le \lambda_i^2 \max_{m\in M}\mathrm{e}^{2u}$. The other inequality can be proven in a completely analogous way. □

4. Loewner's inequality

Proposition 4.1. *Let g be any Riemannian metric on T^2 and let χ be a non-trivial spin structure. There is a flat metric g_0 which is conformal to g:*

(a) $$\frac{\mathrm{sys}_1(T^2, g)^2}{\mathrm{area}(T^2, g)} \le \frac{\mathrm{sys}_1(T^2, g_0)^2}{\mathrm{area}(T^2, g_0)} \quad (\textit{Loewner's inequality}),$$

(b) $$\frac{\mathrm{sys}_1(T^2, g_0)^2}{\mathrm{area}(T^2, g_0)} = \sigma_1(T^2, g_0)^2 = \sigma_1(T^2, g)^2,$$

(c) $$\frac{\text{spin-sys}_1(T^2, g_0, \chi)^2}{\mathrm{area}(T^2, g_0)} \ge \frac{1}{4\|\chi\|^2_{L^2(T^2, g_0)}} = \frac{1}{4\|\chi\|^2_{L^2(T^2, g)}}.$$

We have equality in the inequalities of (a) if and only if g is flat.

For the characterization of the equality case in (c) we choose a lattice Γ together with an isometry $I : \mathbb{R}^2/\Gamma \to (T^2, g_0)$. Then equality in (c) is equivalent to the fact that there are generators γ_1, γ_2 for the lattice Γ satisfying $\gamma_1 \perp \gamma_2$, $\chi(I \circ \gamma_1) = 1$ and $\chi(I \circ \gamma_2) = -1$.

Proof. We follow [8, 4.1]. Let $g = \mathrm{e}^{2u} g_0$. We start with a non-contractible loop c which is shortest with respect to g_0. There is an isometric torus action on (T^2, g_0) acting by translations. Translation by $x \in T^2$ will be denoted by L_x. Then

$$\int_{T^2, g_0} \mathrm{d}x \, \mathrm{length}_g(L_x(c)) = \mathrm{sys}_1(T^2, g_0) \int_{T^2, g_0} \mathrm{d}x \, \mathrm{e}^{u(x)} \le \mathrm{sys}_1(T^2, g_0)\mathrm{area}(T^2, g_0)^{1/2}\mathrm{area}(T^2, g)^{1/2}.$$

Because the left hand side is an upper bound for $\mathrm{sys}_1(T^2, g)\mathrm{area}(T^2, g_0)$, inequality (a) follows.

The discussion of the equality case in (a) is straightforward; (b) and (c) follow directly from elementary calculations. As already stated previously, the L^2-norm is invariant under conformal changes. □

In Corollary 2.3 we also use the following lemma. The proof of it is straightforward.

Lemma 4.2.

$$\mathrm{e}^{2\,\mathrm{osc}\, u} \frac{\text{spin-sys}_1(T^2, g, \chi)^2}{\mathrm{area}(T^2, g)} \geq \frac{\text{spin-sys}_1(T^2, g_0, \chi)^2}{\mathrm{area}(T^2, g_0)}.$$

5. Spectra of flat two-tori

In this section we recall the well-known formula for the spectrum of the Dirac operator on flat two-tori. We restrict to the case that T^2 carries a non-trivial spin structure.

Definition 5.1. The *spin-conformal moduli space* $\mathcal{M}^{\mathrm{spin}}$ is the set of all $(x, y) \in \mathbb{R}^2$ satisfying

$$0 \leq x \leq \tfrac{1}{2}, \qquad (x - \tfrac{1}{2})^2 + y^2 \geq \tfrac{1}{4}, \qquad y > 0. \tag{6}$$

For any $(x, y) \in \mathcal{M}^{\mathrm{spin}}$ we obtain a flat two-torus carrying a non-trivial spin structure as follows:

$$T^2 = \frac{\mathbb{R}^2}{\Gamma_{xy}}, \qquad \Gamma_{xy} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\}.$$

The spin structure $\chi \in H^1(T^2, \mathbb{Z}_2)$ is characterized by

$$\chi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1, \qquad \chi \begin{pmatrix} x \\ y \end{pmatrix} = -1.$$

Conversely any flat torus with a non-trivial spin structure can be rescaled to a torus obtained from $\mathcal{M}^{\mathrm{spin}}$. The dual lattice $\Gamma^*_{xy} := H^1(T^2, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(\Gamma_{xy}, \mathbb{Z})$ is generated by the vectors

$$e_1 := \begin{pmatrix} 1 \\ -\dfrac{x}{y} \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} 0 \\ \dfrac{1}{y} \end{pmatrix}.$$

$$\chi = \left[\frac{1}{2} e_2 \right] \in \frac{(1/2)\Gamma^*_{xy}}{\Gamma^*_{xy}}.$$

Proposition 5.2. ([7]). *Assume that T^2 carries a non-trivial spin structure. Then with the above notations the spectrum of the square of the Dirac operator D^2 on T^2 is given by*

$$\frac{4\pi^2\|\gamma\|^2_{L^2}}{\text{area}},$$

where for each $\gamma \in \Gamma^*_{xy} + (e_2/2)$ we obtain an eigenspace of dimension 2.

Proof. Let (ψ_1, ψ_2) be a basis of parallel sections of the spinor bundle on $\mathbb{R}^2$ and assume that they are pointwise orthogonal. Then

$$\Psi_{j,\gamma} := \exp(2\pi i\langle\gamma, x\rangle)\,\psi_j, \qquad \gamma \in \Gamma^*_{xy} + \tfrac{1}{2}e_2$$

is a spinor field that is invariant under the action of Γ_{xy}. Thus, it defines an eigenspinor for $D^2 : \Sigma T^2 \to \Sigma T^2$ with eigenvalue $4\pi^2|\gamma|^2$ and the family $(\Psi_{j,\gamma}|j = 1, 2; \gamma \in \Gamma^*_{xy} + (e_2/2))$ is a complete system of eigenspinors. □

We want to prove that $\Gamma^*_{xy} + (e_2/2)$ contains no vector that is shorter than $e_2/2$. For this we need a lemma.

Lemma 5.3. *If linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$ satisfy*

$$0 \le \langle v_1, v_2\rangle \le |v_1|^2 \le |v_2|^2,$$

then for any integers a, b with $a \ne 0$ and $b \ne 0$ the following inequality holds

$$|av_1 + bv_2| \ge |v_2 - v_1|.$$

If $|av_1 + bv_2| = |v_2 - v_1|$, then $|a| = |b| = 1$.

Proof. Let $|av_1 + bv_2| \le |v_2 - v_1|$. Without loss of generality we can assume that a and b are relatively prime.

We obtain

$$a^2|v_1|^2 - 2|ab|\cdot\langle v_1, v_2\rangle + b^2|v_2|^2 \le |v_1|^2 - 2\langle v_1, v_2\rangle + |v_2|^2,$$

and therefore

$$\begin{aligned}(a^2 + b^2 - 2)|v_1|^2 &\le (a^2 - 1)|v_1|^2 + (b^2 - 1)|v_2|^2\\ &\le 2(|ab| - 1)\langle v_1, v_2\rangle \le 2(|ab| - 1)|v_1|^2.\end{aligned}$$

Thus $(|a| - |b|)^2 \le 0$ holds, i.e. $|a| = |b|$, and as we assumed that a and b are relatively prime we obtain $|a| = |b| = 1$. Because of $|v_1 + v_2| \ge |v_2 - v_1|$ the lemma holds. □

Corollary 5.4. *If $(x, y) \in \mathcal{M}^{\text{spin}}$, then*:

(a) *There is no vector in $\Gamma^*_{xy} + (e_2/2)$ that is shorter than $e_2/2$.*

(b) *The shortest vectors in* $\Gamma^*_{xy} - \{0\}$ *have length*

$$\min\left\{\frac{1}{y}, \frac{\sqrt{x^2+y^2}}{y}\right\}.$$

Proof.

(a) Because of relations (6) the vectors $v_1 := e_1/2$ and $v_2 := (e_1+e_2)/2$ satisfy the conditions of the lemma. Any element γ of $\Gamma^*_{xy} + (e_2/2)$ can be written as $av_1 + bv_2$, $a, b \in \mathbb{Z} - \{0\}$. The lemma yields

$$|\gamma| \geq |v_2 - v_1| = \tfrac{1}{2}|e_2|.$$

(b) This time we set $v_1 = e_1$ and $v_2 = e_1 + e_2$. As before $0 \leq \langle v_1, v_2\rangle \leq |v_1|^2 \leq |v_2|^2$. Any $\gamma \in \Gamma^*_{xy} - \{0\}$ is either a multiple of v_1 or v_2 (then $|\gamma|^2 \geq |v_1|^2 = |e_1|^2 = 1 + (x^2/y^2)$) or

$$|\gamma| \geq |v_2 - v_1| = \frac{1}{y}. \qquad \square$$

Using area $= y$ we see that the smallest positive eigenvalue λ_1 of D satisfies

$$\lambda_1^2 \text{area} = \frac{\pi^2}{y} = 4\pi^2\|\chi\|^2_{L^2}.$$

Also note for the cosystole

$$\sigma_1^2 = \min\left\{\frac{1}{y}, \frac{x^2+y^2}{y}\right\}.$$

6. Controlling the conformal scaling function

Let T^2 carry an arbitrary metric g. According to the uniformization theorem we can write $g = \mathrm{e}^{2u}g_0$ with a real function $u : T^2 \to \mathbb{R}$ and a flat metric g_0. The function u is unique up to adding a constant.

The aim of this section is to estimate the quantity $\operatorname{osc} u := \max u - \min u$. This estimate will be a Sobolev type estimate. However, as we are interested in an explicit bound, we will use elementary methods for the proof.

Theorem 6.1. *We assume*

$$\|K_g\|_{L^1(T^2,g)} < 4\pi.$$

Then for any $p > 1$ *we obtain a bound for the oscillation of u*

$$\operatorname{osc} u \leq \mathcal{S}_p(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}(\text{area}(T^2,g))^{1-(1/p)}, \sigma_1(T^2,g)^{-2}),$$

where $\mathcal{S}$ *is the function defined in* Definition 2.1. *Equality is obtained if and only if g is flat.*

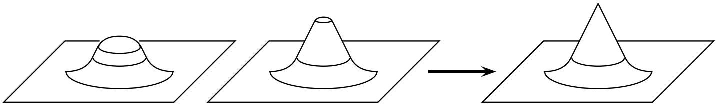

Fig. 1. Metrics with $\|K_{g_i}\|_{L^1(T^2,g_i)} \le \mathcal{K}_1$ and $\operatorname{osc} u_{g_i} \to \infty$.

Corollary 6.2. *Let $\mathcal{F}$ be a family of Riemannian metrics conformal to the flat metric g_0. Assume that there are constants $\mathcal{K}_1 \in]0, 4\pi[$ and $\mathcal{K}_p \in]0, \infty[$, $p \in]1, \infty[$ with*

$$\|K_g\|_{L^1(T^2,g)} \le \mathcal{K}_1 \quad \text{and} \quad \|K_g\|_{L^p(T^2,g)}(\operatorname{area}(T^2, g))^{1-(1/p)} \le \mathcal{K}_p \text{ for any } g \in \mathcal{F}.$$

Then the oscillation $\operatorname{osc} u_g$ of the scaling function corresponding to g is uniformly bounded on $\mathcal{F}$ by

$$\operatorname{osc} u_g \le \mathcal{S}(\mathcal{K}_1, \mathcal{K}_p, p, \mathcal{V}(T^2, g_0)).$$

Before proving the theorem we will present some examples showing that the theorem and the corollary no longer hold if we drop one of the assumptions $\|K_g\|_{L^1(T^2,g)} \le \mathcal{K}_1 < 4\pi$ or $\|K_g\|_{L^p(T^2,g)}(\operatorname{area}(T^2, g))^{1-(1/p)} \le \mathcal{K}_p$.

Example. For any $\mathcal{K}_1 > 0$ there is a sequence (g_i) of Riemannian metrics with fixed conformal type, bounded volume, constant systole, with

$$\|K_{g_i}\|_{L^1(T^2,g_i)} \le \mathcal{K}_1 \text{ and } \operatorname{osc} u_{g_i} \to \infty.$$

In order to construct such a sequence we take a flat torus and replace a ball by a rotationally symmetric surface which approximates a cone for $i \to \infty$ (Fig. 1).

Example. For any $\varepsilon > 0$ there is a sequence (g_i) of Riemannian metrics with fixed conformal type, bounded volume, constant systole, $-1 \le K_{g_i} \le 1$, $\|K_{g_i}\|_{L^1(T^2,g_i)} \le 4\pi + \varepsilon$, $\|K_{g_i}\|_{L^p(T^2,g_i)} \le$ const and $\operatorname{osc} u_{g_i} \to \infty$. In order to construct such a sequence we take a ball out of a flat torus and replace it by a hyperbolic part, a cone of small opening angle, and a cap as indicated in Fig. 2. While the injectivity radius of the hyperbolic part shrinks to zero, the oscillation of u tends to infinity.

In Fig. 2 the dots in the “limit space” indicate the hyperbolic part with injectivity radius tending to 0 and diameter tending to ∞.

Proof of Theorem 6.1. As Morse functions form a dense subset of the space of C^∞-functions with respect to the C^∞-topology, we can assume without loss of generality that u is a Morse function. We set $\text{Area}_g := \operatorname{area}(T^2, g)$ and $\text{Area}_0 := \operatorname{area}(T^2, g_0)$. We define

$$G_<(v) := \{x \in T^2 \mid u(x) < v\}, \quad G_>(v) := \{x \in T^2 \mid u(x) > v\},$$

$$\varphi : [0, \text{Area}_g] \to \mathbb{R},$$

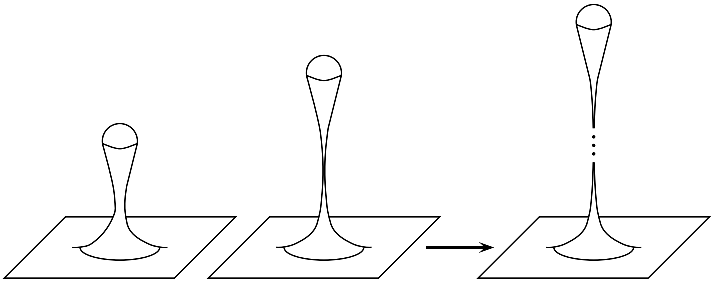

Fig. 2. Metrics without $\mathcal{K}_1 < 4\pi$ and $\operatorname{osc} u_{g_i} \to \infty$.

$$A \mapsto \inf\{\sup_{x\in X} u(x) | X \subset T^2 \text{ open, area}(X) \geq A\}, \tag{7}$$

$$= \sup\{\inf_{x\in X^c} u(x) | X^c \subset T^2 \text{ open, area}(X^c) \geq \text{Area}_g - A\}. \tag{8}$$

The infimum in (7) is actually a minimum and, as u is a Morse function, the only minimum is attained exactly for $\bar{X}^c = G_<(\varphi(A))$. Similarly the supremum in (8) is attained exactly in $\bar{X}^c = G_>(\varphi(A))$. The function φ is strictly increasing and is continuously differentiable (Fig. 3). The inverse of φ is given by

$$\varphi^{-1}(v) = \text{area}(G_<(u)).$$

The differential $\varphi'(A)$ is zero if and only if $\varphi(A)$ is a critical value of u.

Now let $v \in [\min u, \max u]$ be a regular value of u. We obtain

$$(\varphi^{-1})'(v) = \int_{\partial G_<(v),g} \frac{1}{|\mathrm{d}u|_g} \geq \frac{\text{length}(\partial G_<(v), g)^2}{\int_{\partial G_<(v),g} |\mathrm{d}u|_g}, \tag{9}$$

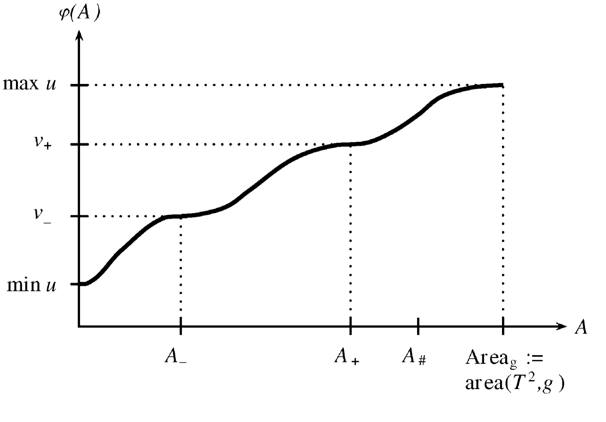

Fig. 3. The function ϕ.

where $\mathrm{length}(\partial G_<(v), g)$ is the length of the boundary of $\partial G_<(v)$ with respect to g. This inequality will yield an upper bound for φ' which will provide in turn an upper bound for $\operatorname{osc} u = \varphi(\mathrm{Area}_g) - \varphi(0) = \int_0^{\mathrm{Area}_g} \varphi'$. We transform

$$\int_{\partial G_<(v),g} |\mathrm{d}u|_g = \int_{\partial G_<(v)} * \mathrm{d}u = -\int_{G_<(v),g} \Delta_g u = -\int_{G_<(v),g} K_g. \tag{10}$$

The last equation follows from the Kazdan–Warner equation $\Delta_g u = K_g$ [14]. We define κ using the Gaussian curvature function $K_g : T^2 \to \mathbb{R}$

$$\kappa : [0, \mathrm{Area}_g] \to \mathbb{R}, \quad \kappa(A) := \inf\left\{\sup_{x\in X} K_g(x) | X \subset T^2 \text{ open},\ \mathrm{area}(X) \geq A\right\}.$$

Any open subset $X \subset T^2$ satisfies

$$\int_0^{\mathrm{area}(X,g)} \kappa \leq \int_{X,g} K_g \leq \int_{\mathrm{Area}_g - \mathrm{area}(X,g)}^{\mathrm{Area}_g} \kappa$$

and for $X = T^2$ we have equality. Using Gauss–Bonnet theorem we see that $\int_0^{\mathrm{Area}_g} \kappa = 0$. The right hand side of Eq. (10) now can be estimated as follows:

$$-\int_{G_<(\varphi(A)),g} K_g \leq -\int_0^A \kappa = \int_A^{\mathrm{Area}_g} \kappa. \tag{11}$$

Putting (9)–(11) together, we obtain

$$\varphi'(A) \leq \frac{\int_A^{\mathrm{Area}_g} \kappa}{\mathrm{length}(\partial G_<(\varphi(A)), g)^2}.$$

Our next goal is to find suitable lower bounds for $\mathrm{length}(\partial G_<(\varphi(A))$.

Note that for any regular value v of u, we can apply the following lemma for $X_1 = G_<(v)$ and $X_2 = G_>(v)$. □

Lemma 6.3. *Let (X_1, X_2) be two disjoint open subsets of T^2 such that they have a common smooth boundary $\partial X_1 = \partial X_2$. Then exactly one of the following conditions is satisfied*:

(i) *The inclusion $X_1 \to T^2$ induces the trivial map $\pi_1(X_1) \to \pi_1(T^2)$.*
(ii) *The inclusion $X_2 \to T^2$ induces the trivial map $\pi_1(X_2) \to \pi_1(T^2)$.*
(iii) *The boundary ∂X_1 has at least two components that are non-contractible in T^2.*

Before proving the lemma we continue with the proof of Theorem 6.1.

If condition (i) is satisfied by v, it is obvious that it is also satisfied by $v' \in [0, v]$. Similarly, if condition (ii) is satisfied by v, then it is also satisfied by $v' \in [v, \mathrm{Area}_g]$

$$v_- := \sup\{v \in [0, \mathrm{Area}_g] | \text{(i) is satisfied for } v\},$$
$$v_+ := \inf\{v \in [0, \mathrm{Area}_g] | \text{(ii) is satisfied for } v\}, \qquad A_\pm := \varphi^{-1}(v_\pm).$$

In each of the three cases we derive a different estimate for $\mathrm{length}(\partial G_<(v), g)$ and therefore we obtain a different bound for φ'.

(i) In this case $G_<(v)$ can be lifted to the universal covering $\mathbb{R}^2$ of T^2. We will also write g and g_0 for the pullbacks of g and g_0 to $\mathbb{R}^2$. The isoperimetric inequality of the flat space $(\mathbb{R}^2, g_0)$ yields

$$\text{length}(\partial G_<(v), g_0)^2 \geq 4\pi\,\text{area}(G_<(v), g_0).$$

Using the relations

$$\text{length}(\partial G_<(v), g) = \mathrm{e}^v \text{length}(\partial G_<(v), g_0), \tag{12}$$

$$\text{area}(G_<(v), g) \leq \mathrm{e}^{2v}\,\text{area}(G_<(v), g_0), \tag{13}$$

we obtain

$$\text{length}(\partial G_<(v), g)^2 \geq 4\pi\,\text{area}(G_<(v), g). \tag{14}$$

Together with the Hölder inequality $-\int_0^A \kappa \leq \|K_g\|_{L^p(T^2,g)} A^{1-(1/p)}$ we get

$$\varphi'(A) = \frac{1}{(\varphi^{-1})'(\varphi(A))} \leq \frac{-\int_0^A \kappa}{\text{length}(\partial G_<(\varphi(A)), g)^2} \leq \frac{1}{4\pi}\|K_g\|_{L^p(T^2,g)} A^{-1/p}.$$

Integration yields

$$\begin{aligned} v_- - \min u = \varphi(\varphi^{-1}(v_-)) - \varphi(0) &\leq \frac{p}{p-1}\frac{1}{4\pi}\|K_g\|_{L^p(T^2,g)} (\varphi^{-1}(v_-))^{1-(1/p)} \\ &\leq \frac{p}{p-1}\frac{1}{4\pi}\|K_g\|_{L^p(T^2,g)} (\text{Area}_g)^{1-(1/p)}. \end{aligned} \tag{15}$$

(ii) This case is similar to the previous one, but unfortunately because of opposite signs some estimates do not work as before. For example, (13) and (14) are no longer true for $G_<(v)$ replaced by $G_>(v)$. Instead we use Topping's inequality [15,16]

$$(\text{length}(\partial G_>(v), g))^2 \geq 4\pi\hat{A} - 2\int_0^{\hat{A}} (\hat{A} - a)\kappa(\text{Area}_g - a)\,\mathrm{d}a \tag{16}$$

with $\hat{A} = \text{area}(G_>(v), g)$. Using the estimate

$$\int_0^{\hat{A}} (\hat{A}-a)\kappa(\text{Area}_g - a)\,\mathrm{d}a \leq \hat{A}\int_0^{\hat{A}} \max\{0, \kappa\text{Area}_g - a)\}\,\mathrm{d}a \leq \frac{\hat{A}}{2}\|K_g\|_{L^1(T^2,g)},$$

we obtain

$$(\text{length}(\partial G_>(v), g))^2 \geq (4\pi - \|K_g\|_{L^1(T^2,g)})\hat{A}. \tag{17}$$

The obvious inequality $\int_{\text{Area}_g-\hat{A}}^{\text{Area}_g} \kappa \leq \|\max\{0, K_g\}\|_{L^1(T^2,g)} \leq (1/2)\|K_g\|_{L^1(T^2,g)}$ yields

$$\varphi'(\text{Area}_g - \hat{A}) \leq \frac{1}{\hat{A}}\frac{\|K_g\|_{L^1(T^2,g)}}{8\pi - 2\|K_g\|_{L^1(T^2,g)}}.$$

After integration we have

$$\varphi(\mathrm{Area}_g - \hat{A}) - \varphi(A_+) \leq \log\left(\frac{\mathrm{Area}_g - A_+}{\hat{A}}\right)\frac{\|K_g\|_{L^1(T^2,g)}}{8\pi - 2\|K_g\|_{L^1(T^2,g)}}.$$

The right hand side converges to ∞ for $\hat{A} \to 0$. Thus we have to improve our estimates for small $\hat{A}$. The integral in (16) also has the following bound:

$$\begin{aligned}
&\int_0^{\hat{A}} (\hat{A} - a)\kappa(\mathrm{Area}_g - a)\,\mathrm{d}a \\
&\quad \leq \left(\int_0^{\hat{A}} (\hat{A} - a)^q\,\mathrm{d}a\right)^{1/q} \cdot \left(\int_0^{\hat{A}} |\kappa(\mathrm{Area}_g - a)|^p\,\mathrm{d}a\right)^{1/p} \\
&\quad = \left(\frac{\hat{A}^{q+1}}{q+1}\right)^{1/q} \cdot \|K_g\|_{L^p(T^2,g)},
\end{aligned} \tag{18}$$

where we wrote $q := p/(p-1)$ in order to simplify the notation.

We obtain a second lower bound on the length

$$(\mathrm{length}(\partial G_>(v), g))^2 \geq 4\pi\hat{A} - c\hat{A}^{1+(1/q)}\|K_g\|_{L^p(T^2,g)} \tag{19}$$

for any $c \geq 2/\sqrt[q]{q+1}$, e.g. $c = 2$. Note that our assumption $\|K_g\|_{L^1(T^2,g)} < 4\pi$ does not imply that the right hand side of the above inequality is always positive. Although (19) is better for small $\hat{A}$, it is not strong enough to control the length for larger $\hat{A}$. However, for

$$\hat{A} < \left(\frac{4\pi}{c \cdot \|K_g\|_{L^p(T^2,g)}}\right)^q,$$

we use (19) and

$$\int_{\mathrm{Area}_g - \hat{A}}^{\mathrm{Area}_g} \kappa \leq \hat{A}^{1/q}\|K_g\|_{L^p(T^2,g)},$$

to obtain the estimate

$$\varphi'(\mathrm{Area}_g - \hat{A}) \leq \frac{\hat{A}^{-1/p}\|K_g\|_{L^p(T^2,g)}}{4\pi - c\hat{A}^{1/q}\|K_g\|_{L^p(T^2,g)}}.$$

With the substitution

$$w = w(A) = 4\pi - c(\mathrm{Area}_g - A)^{1/q}\|K_g\|_{L^p(T^2,g)},$$

integration yields

$$\begin{aligned}
\varphi(\mathrm{Area}_g) - \varphi(A_\#) &= \int_{A_\#}^{\mathrm{Area}_g} \varphi'(A)\,\mathrm{d}A \leq \int_{w(A_\#)}^{w(\mathrm{Area}_g)} \frac{q}{c}\frac{1}{w}\,\mathrm{d}w = \frac{q}{c}\log\frac{w(\mathrm{Area}_g)}{w(A_\#)} \\
&= \frac{q}{c}\log\frac{4\pi}{4\pi - c(\mathrm{Area}_g - A_\#)^{1/q}\|K_g\|_{L^p(T^2,g)}}
\end{aligned}$$

for any $A_\#$ between $\mathrm{Area}_g - (4\pi/(c \cdot \|K_g\|_{L^p(R^2,g)}))^q$ and Area_g. We choose

$$A_\# := \max\left\{\mathrm{Area}_g - \left(\frac{\|K_g\|_{L^1(T^2,g)}}{2\|K_g\|_{L^p(T^2,g)}}\right)^q, A_+\right\}.$$

Finally we obtain the estimates

$$\max u - \varphi(A_\#) \le \frac{q}{c}\log\frac{8\pi}{8\pi - c\|K_g\|_{L^p(T^2,g)}}, \tag{20}$$

$$\varphi(A_\#) - v_+ \le q\frac{\|K_g\|_{L^1(T^2,g)}}{8\pi - 2\|K_g\|_{L^1(T^2,g)}}\log\left(\frac{2\mathrm{Area}_g^{1/q}\|K_g\|_{L^p(T^2,g)}}{\|K_g\|_{L^1(T^2,g)}}\right). \tag{21}$$

For $c = 2$ the right hand sides of these inequalities contribute two summands to the formula for $\mathcal{S}$.

(iii) If $v = \varphi(A)$ is a regular value of u between v_- and v_+, then $\partial G_<(v)$ contains at least two components that are non-contractible in T^2. Hence, for any metric $\tilde{g}$ on T^2 we get

$$\mathrm{length}(\partial G_<(v), \tilde{g}) \ge 2\mathrm{sys}_1(T^2, \tilde{g}).$$

In order to prove (a) of Theorem 6.1 we apply this equation to $\tilde{g} := g_0$. Using $\int_A^{\mathrm{Area}_g}\kappa \le (1/2)\|K_g\|_{L^1(T^2,g)}$ and $\mathrm{length}(\partial G_<(v), g) = e^v\,\mathrm{length}(\partial G_<(v), g_0)$ we obtain

$$\varphi'(A) \le \mathrm{e}^{-2\varphi(A)}\frac{\int_A^{\mathrm{Area}_g}\kappa}{4\mathrm{sys}_1(T^2, g_0)^2} \le \frac{1}{8}\mathrm{e}^{-2\varphi(A)}\frac{\|K_g\|_{L^1(T^2,g)}}{\mathrm{sys}_1(T^2, g_0)^2}. \tag{22}$$

Integration yields

$$\begin{aligned} v_+ - v_- &= \int_{A_-}^{A_+}\varphi'(A)\,\mathrm{d}A \le \frac{1}{8}\frac{\|K_g\|_{L^1(T^2,g)}}{\mathrm{sys}_1(T^2, g_0)^2}\int_{A_-}^{A_+}\mathrm{e}^{-2\varphi(A)}\,\mathrm{d}A \\ &\le \frac{1}{8}\frac{\|K_g\|_{L^1(T^2,g)}}{\mathrm{sys}_1(T^2, g_0)^2}\mathrm{Area}_0, \end{aligned} \tag{23}$$

where we used $\mathrm{Area}_0 = \mathrm{area}(T^2, g_0) = \int_0^{\mathrm{Area}_g}\mathrm{e}^{-2\varphi(A)}\,\mathrm{d}A$.

Together with inequalities (15), (20) and (21) we obtain the statement of the theorem.

Proof of Lemma 6.3. Assume that (X_1, X_2) satisfies (iii), then ∂X_1 contains a non-contractible loop. By a small perturbation we can achieve that this loop lies completely in X_1. Therefore $\pi_1(X_1) \to \pi_1(T^2)$ is not trivial. Hence (X_1, X_2) does not satisfy (i). Similarly we prove that it does not satisfy (ii).

Now assume that (X_1, X_2) satisfies both (i) and (ii). Van-Kampen's theorem implies $\pi_1(T^2) = 0$. Therefore we have shown that at most one of the three conditions is satisfied.

It remains to show that at least one condition is satisfied. For this we assume that neither (i) nor (ii) is satisfied, i.e. there are continuous paths $c_i : S^1 \to X_i$ that are non-contractible within T^2. Obviously ∂X_1 is homologous to zero. We will show that at least one component of ∂X_1 is non-homologous to zero. Then there has to be a second component that is non-homologous to zero, because $[\partial X_1] = 0$ is the sum of the homology classes of the components.

We argue by contradiction. Assume that each component of ∂X_1 is homologous to zero. Let $\pi : \mathbb{R}^2 \to T^2$ be the universal covering. Then $\pi^{-1}(\partial X_1)$ is diffeomorphic to a disjoint union of countably many S^1. We write

$$\pi^{-1}(\partial X_1) = \dot{\bigcup_{i\in\mathbb{N}}} Y_i$$

with $Y_i \cong S^1$. We choose lifts $\tilde{c}_i : \mathbb{R} \to \mathbb{R}^2$ of c_i, i.e. $\pi(\tilde{c}_i(t+z)) = c_i(t)$ for all $t \in [0,1]$, $z \in \mathbb{Z}$ and $i = 1, 2$. Then we take a path $\tilde{\gamma} : [0,1] \to \mathbb{R}^2$ joining $\tilde{c}_1(0)$ to $\tilde{c}_2(0)$. We can assume that $\tilde{\gamma}$ is transversal to any Y_i. We define I to be the set of all $i \in \mathbb{N}$ such that Y_i meets the trace of $\tilde{\gamma}$. The set I is finite. Using the theorem of Jordan and Schoenfliess about simple closed curves in $\mathbb{R}^2$ we can inductively construct a compact set $K \subset \mathbb{R}^2$ with boundary $\cup_{i\in I} Y_i$. The number of intersections of $\tilde{\gamma}$ with $\cup_{i\in I} Y_i$ is odd. Thus, either $\tilde{c}_1(0)$ or $\tilde{c}_2(0)$ is in the interior of K. But if $\tilde{c}_i(0)$ is in the interior of K, then the whole trace $\tilde{c}_i(\mathbb{R})$ is contained in K. Furthermore, $\tilde{c}_i(\mathbb{R}) = \pi^{-1}(c_i([0,1]))$ is closed and therefore compact. This implies that c_i is homologous to zero in contradiction to our assumption. □

Acknowledgements

The author wants to thank Christian Bär for many interesting and stimulating discussions about the subject.

References

[1] B. Ammann, C. Bär, Dirac eigenvalue estimates on surfaces, Math. Z. 240 (2002) 423–449
[2] B. Ammann, Spectral estimates on 2-tori. http://arxiv.org/abs/math.dg/0101061.
[3] B. Ammann, Spin-Strukturen und das Spektrum des Dirac-Operators, Ph.D. Thesis, University of Freiburg, Germany, 1998, Shaker-Verlag, Aachen, 1998. ISBN 3-8265-4282-7.
[4] C. Bär, Extrinsic bounds for eigenvalues of the Dirac operator, Ann. Glob. Anal. Geom. 16 (1998) 573–596.
[5] H. Baum, Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten, Teubner Verlag, 1981.
[6] T. Friedrich, Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannig-faltigkeit nicht-negativer Krümmung, Math. Nach. 97 (1980) 117–146.
[7] T. Friedrich, Zur Abhängigkeit des Dirac-Operators von der Spin-Struktur, Colloq. Math. 48 (1984) 57–62.
[8] M. Gromov, Structures métriques pour les variétés Riemanniennes, CEDIC, Paris, 1981.
[9] O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and killing spinors, Commun. Math. Phys. 104 (1986) 151–162.
[10] N. Hitchin, Harmonic spinors, Adv. Math. 14 (1974) 1–55.
[11] K.-D. Kirchberg, An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature, Ann. Glob. Anal. Geom. 4 (1986) 291–325.
[12] K.-D. Kirchberg, Compact six-dimensional Kähler spin manifolds of positive scalar curvature with the smallest possible first eigenvalue of the Dirac operator, Math. Ann. 282 (1988) 157–176.
[13] W. Kramer, U. Semmelmann, G. Weingart, Eigenvalue estimates for the Dirac operator on quater-nionic Kähler manifolds, Math. Z. 230 (1999) 727–751.
[14] J.L. Kazdan, F.W. Warner, Curvature functions for compact 2-manifolds, Ann. Math. 99 (1974) 14–47.
[15] P. Topping, Mean curvature flow and geometric inequalities, J. Reine Angew. Math. 503 (1998) 47–61.
[16] P. Topping, The isoperimetric inequality on a surface, Manuscripta Math. 100 (1999) 23–33.